

# Amenability and Unique Ergodicity of Automorphism Groups of Fraïssé Structures

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## Abstract

In this paper we provide a necessary and sufficient condition for the amenability of the automorphism group of Fraïssé structures and apply it to prove the non-amenability of the automorphism groups of the directed graph  $\mathbf{S}(3)$  and the Boron tree structure  $\mathbf{T}$ . Also, we provide a negative answer to the Unique Ergodicity-Generic Point problem of Angel-Kechris-Lyons [AKL]. By considering  $\mathrm{GL}(\mathbf{V}_\infty)$ , where  $\mathbf{V}_\infty$  is the countably infinite dimensional vector space over a finite field  $F_q$ , we show that the unique invariant measure on the universal minimal flow of  $\mathrm{GL}(\mathbf{V}_\infty)$  is not supported on the generic orbit.

## 1 Introduction

Let  $G$  be a Hausdorff topological group. A  $G$ -**flow**  $X$  is a compact Hausdorff space  $X$  along with a continuous  $G$ -action; it is **minimal** if every orbit is dense. A quick application of Zorn's lemma shows that every  $G$ -flow contains a minimal subflow. A  $G$ -flow  $X$  is **universal** if there is a surjective homomorphism from  $X$  to any other minimal  $G$ -flow  $Y$ , where a homomorphism of  $G$ -flows is just a continuous map which respects the  $G$  action. It is a fact that every such  $G$  has a universal minimal flow  $M(G)$  which is unique up to isomorphism. The universal minimal flow is extremely useful for characterizing  $G$ -flows. For example, if  $M(G)$  is a singleton, we say that  $G$  is **extremely amenable**; every  $G$ -flow has a fixed point. A group  $G$  is **amenable** if every  $G$ -flow supports an invariant Borel probability measure; by push forward, we see that this is the case iff  $M(G)$  supports such a measure. We say  $G$  is **uniquely ergodic** if every  $G$ -flow supports a unique such measure; indeed, it is a fact that this occurs iff  $M(G)$  supports a unique such measure (see [AKL]). Given a  $G$ -flow  $X$ , we say that  $x \in X$  is **generic** if the orbit  $G \cdot x$  is comeager; note that if there is a comeager orbit, it is unique. The group  $G$  has the **generic point property** if every minimal  $G$ -flow has a generic point, and once again, this is equivalent to  $M(G)$  having a generic point. However, computing  $M(G)$  is

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normally quite difficult; indeed,  $M(G)$  is often not metrizable, as is the case whenever  $G$  is countably infinite and discrete. However, there are many situations where  $M(G)$  can be computed explicitly and is metrizable for  $G$  the automorphism group of a Fraïssé structure.

A **language**  $L = \{R_i\}_{i \in I} \cup \{f_j\}_{j \in J} \cup \{c_k\}_{k \in K}$  is a set of relation, function, and constant symbols. Each relation and function symbol has an arity  $n(i), m(j) \in \mathbb{N}^+$  for  $i \in I, j \in J$ . In this paper, all languages are assumed to be countable. An  $L$ -**structure**  $\mathbf{A} = \langle A, \{R_i^{\mathbf{A}}\}_{i \in I}, \{f_j^{\mathbf{A}}\}_{j \in J}, \{c_k^{\mathbf{A}}\}_{k \in K} \rangle$  consists of a set  $A \neq \emptyset$  called the **universe** of  $\mathbf{A}$ ,  $R_i^{\mathbf{A}} \subseteq A^{n(i)}$ ,  $f_j^{\mathbf{A}} : A^{m(j)} \rightarrow A$ , and  $c_k^{\mathbf{A}} \in A$ . An **embedding/isomorphism**  $\pi : \mathbf{A} \rightarrow \mathbf{B}$  of  $L$ -structures is an injective/bijective map which preserves structure:  $R_i^{\mathbf{A}}(a_1, \dots, a_{n(i)}) \Leftrightarrow R_i^{\mathbf{B}}(\pi(a_1), \dots, \pi(a_{n(i)}))$ ,  $\pi(f_j^{\mathbf{A}}(a_1, \dots, a_{m(j)})) = f_j^{\mathbf{B}}(\pi(a_1), \dots, \pi(a_{m(j)}))$ , and  $\pi(c_k^{\mathbf{A}}) = c_k^{\mathbf{B}}$ . We write  $\mathbf{A} \leq \mathbf{B}$  if there is an embedding from  $\mathbf{A}$  into  $\mathbf{B}$ , and we write  $\mathbf{A} \cong \mathbf{B}$  if there is an isomorphism between the two. We say  $\mathbf{A}$  is a **substructure** of  $\mathbf{B}$ , written  $\mathbf{A} \subseteq \mathbf{B}$ , if  $1_{\mathbf{A}}$  is an embedding of  $\mathbf{A}$  into  $\mathbf{B}$ . If  $L_0 \subseteq L$  and  $\mathbf{A}$  is an  $L$ -structure, we write  $\mathbf{A}|_{L_0}$  for the structure obtained by taking  $\mathbf{A}$  and ignoring the interpretations of relation, function, and constant symbols in  $L \setminus L_0$ . Similarly, if  $\mathcal{K}$  is a class of  $L$ -structures, we write  $\mathcal{K}|_{L_0}$  for the class  $\{\mathbf{A}|_{L_0} : \mathbf{A} \in \mathcal{K}\}$ .

Let  $\mathcal{K}$  be a class of finite  $L$ -structures closed under isomorphism with countably many isomorphism types and such that there are structures in  $\mathcal{K}$  of arbitrarily large finite cardinality. We call  $\mathcal{K}$  a **Fraïssé class** if the following three items hold:

- Hereditary Property (HP): If  $\mathbf{B} \in \mathcal{K}$  and  $\mathbf{A} \leq \mathbf{B}$ , then  $\mathbf{A} \in \mathcal{K}$ .
- Joint Embedding Property (JEP): If  $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ , then there is  $\mathbf{C} \in \mathcal{K}$  such that  $\mathbf{A}, \mathbf{B} \leq \mathbf{C}$ .
- Amalgamation Property (AP): If  $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$  and  $f : \mathbf{A} \rightarrow \mathbf{B}$ ,  $g : \mathbf{A} \rightarrow \mathbf{C}$  are embeddings, then there is  $\mathbf{D} \in \mathcal{K}$  and embeddings  $r : \mathbf{B} \rightarrow \mathbf{D}$ ,  $s : \mathbf{C} \rightarrow \mathbf{D}$  such that  $r \circ f = s \circ g$ .

A **Fraïssé structure**  $\mathbf{K}$  is a countably infinite, locally finite (finitely generated substructures are finite)  $L$ -structure which is **ultrahomogeneous**, i.e. any isomorphism between finite substructures extends to an automorphism of  $\mathbf{K}$ . An equivalent definition is that a Fraïssé structure satisfies the **Extension Property**: let  $\mathbf{A} \subseteq \mathbf{B}$  be finite substructures of  $\mathbf{K}$  and  $f : \mathbf{A} \rightarrow \mathbf{K}$  be an embedding. Then there is an embedding  $g : \mathbf{B} \rightarrow \mathbf{K}$  extending  $f$ . For any infinite structure  $\mathbf{X}$ , let  $\text{Age}(\mathbf{X})$  denote the class of finite substructures of  $\mathbf{X}$ . Fraïssé's theorem states that there is a one-to-one correspondence between Fraïssé classes and Fraïssé structures: the age of each Fraïssé structure is a Fraïssé class, and each Fraïssé class is the age of a Fraïssé structure unique up to isomorphism. For  $\mathcal{K}$  such a class, we write  $\text{Flim}(\mathcal{K})$  for the associated structure, called the **Fraïssé limit** of  $\mathcal{K}$ . See Hodges [Ho] for a more detailed exposition.

For finite structures  $\mathbf{A} \leq \mathbf{B}$ , let  $\binom{\mathbf{B}}{\mathbf{A}}$  denote those substructures of  $\mathbf{B}$  which are isomorphic to  $\mathbf{A}$ . A class  $\mathcal{K}$  of finite structures satisfies the **Ramsey Property** (RP) if for any  $\mathbf{A} \leq \mathbf{B} \in \mathcal{K}$  and any  $k \geq 2$ , there is  $\mathbf{C} \in \mathcal{K}$ ,  $\mathbf{B} \leq \mathbf{C}$  such that for any coloring  $c : \binom{\mathbf{C}}{\mathbf{A}} \rightarrow k$ , there is  $\mathbf{B}_0 \in \binom{\mathbf{C}}{\mathbf{B}}$  for which  $c$  is constant on  $\binom{\mathbf{B}_0}{\mathbf{A}}$ . In 2005, Kechris, Pestov, and Todorcevic [KPT] showed that for a Fraïssé structure  $\mathbf{K}$ ,  $\text{Age}(\mathbf{K})$  has the RP and

consists of rigid structures (having no non-trivial automorphisms) if and only if  $\text{Aut}(\mathbf{K})$  is extremely amenable.

Let  $\mathcal{K}$  and  $\mathcal{K}^*$  be Fraïssé classes with limits  $\mathbf{K}, \mathbf{K}^*$  in languages  $L$  and  $L^* = L \cup \{S_1, \dots, S_n\}$ , with each  $S_i$  a relation not in  $L$ , such that  $\mathcal{K}^*|_L = \mathcal{K}$ . We say the pair  $(\mathcal{K}, \mathcal{K}^*)$  is **reasonable** if for any  $\mathbf{A}^* \in \mathcal{K}^*$ ,  $\mathbf{B} \in \mathcal{K}$ , and embedding  $f : \mathbf{A}^*|_L \rightarrow \mathbf{B}$ , there is  $\mathbf{B}^* \in \mathcal{K}^*$  with  $\mathbf{B}^*|_L = \mathbf{B}$  and  $f : \mathbf{A}^* \rightarrow \mathbf{B}^*$  also an embedding, or equivalently, if  $\mathbf{K}^*|_L = \mathbf{K}$ . In this case,  $\text{Aut}(\mathbf{K})$  acts on the compact, metrizable space  $X_{\mathcal{K}^*}$  of all relations  $(S_1, \dots, S_n)$  on  $\mathbf{K}$  with  $\text{Age}(\langle \mathbf{K}, S_1, \dots, S_n \rangle) \subseteq \mathcal{K}^*$ . The basic neighborhoods of  $X_{\mathcal{K}^*}$  are given by open sets of the form  $N_{\langle \mathbf{A}, T_1, \dots, T_n \rangle} := \{(S_1, \dots, S_n) \in X_{\mathcal{K}^*} : S_i|_{\mathbf{A}} = T_i, 1 \leq i \leq n\}$  for  $\mathbf{A} \subseteq \mathbf{K}$ ,  $\mathbf{A} \in \mathcal{K}$ , and  $\langle \mathbf{A}, T_1, \dots, T_n \rangle \in \mathcal{K}^*$ . The pair satisfies the **Expansion Property** if for each  $\mathbf{A} \in \mathcal{K}$ , there is  $\mathbf{B} \in \mathcal{K}$  such that for any  $\mathbf{A}^*, \mathbf{B}^* \in \mathcal{K}^*$  with  $\mathbf{A}^*|_L = \mathbf{A}$ ,  $\mathbf{B}^*|_L = \mathbf{B}$ , we have  $\mathbf{A}^* \leq \mathbf{B}^*$ . Pairs which are reasonable, satisfy the Expansion Property, and with  $\mathcal{K}^*$  satisfying the RP are called **excellent**. In this case we call  $\mathcal{K}^*$  a **companion** of  $\mathcal{K}$ . In [KPT] (and later in the generality presented here in work of Nguyen Van Thé [LNVNT]), it is shown that given an excellent pair,  $X_{\mathcal{K}^*}$  is the universal minimal flow of  $\text{Aut}(\mathbf{K})$ , from which it follows that a companion class  $\mathcal{K}^*$ , if it exists, is unique in an appropriate sense. Also note that for such  $\mathbf{K}$ ,  $\text{Aut}(\mathbf{K})$  has the generic point property as witnessed by the point  $(S_1, \dots, S_n) \in X_{\mathcal{K}^*}$  with  $\langle \mathbf{K}, S_1, \dots, S_n \rangle = \mathbf{K}^*$ .

Building on ideas in Kechris-Sokić [KS], we will provide a necessary and sufficient condition for  $\text{Aut}(\mathbf{K})$  as above to be amenable; note that  $\text{Aut}(\mathbf{K})$  is amenable iff there is an invariant Borel probability measure on  $X_{\mathcal{K}^*}$ . We will then consider the Fraïssé structures **S**(3) and **T**. The directed graph **S**(3) has vertices points on the unit circle with rational argument; for  $a \neq b$ , we have  $a \rightarrow b$  if  $b$  is less than  $120^\circ$  counterclockwise from  $a$ ; see [LNVNT]. The random Boron tree structure **T** is the Fraïssé limit of the class of finite Boron tree structures. A Boron tree is an unrooted tree where each vertex has degree 1 or 3; any Boron tree  $T$  admits a Boron tree structure  $(T) = \langle L_T, R \rangle$ , where  $L_T$  is the set of leaves of  $T$ , and  $R$  is a 4-ary relation where  $R(a, b, c, d)$  holds if  $a, b, c, d$  are distinct and the path from  $a$  to  $b$  does not intersect the path from  $c$  to  $d$ ; see Jasiński [J]. We will show:

**Theorem 1.1.** *The groups  $\text{Aut}(\mathbf{S}(3))$  and  $\text{Aut}(\mathbf{T})$  are not amenable.*

Consider the case  $L^* = L \cup \{<\}$ , with  $<$  a symbol for a linear ordering. For  $\mathcal{K}$  a class of  $L$ -structures, define  $\mathcal{K} * \mathcal{LO} = \{\langle \mathbf{A}, < \rangle : \mathbf{A} \in \mathcal{K} \text{ and } < \text{ is a linear order on } \mathbf{A}\}$ . Now if  $(\mathcal{K}, \mathcal{K} * \mathcal{LO})$  is an excellent pair, we see that  $\text{Aut}(\mathbf{K})$  is amenable; for  $\mathbf{A} \subseteq \mathbf{K}$ ,  $\mathbf{A} \in \mathcal{K}$  and a basic open neighborhood  $N_{\langle \mathbf{A}, <_0 \rangle} := \{< \in X_{\mathcal{K}^*} : <|_{\mathbf{A}} = <_0\}$  of  $X_{\mathcal{K}^*}$ , an invariant measure is given by  $\mu(N_{\langle \mathbf{A}, <_0 \rangle}) = 1/|A|!$ . This is called the **uniform measure**. It is shown in [AKL] that the uniform measure on  $X_{\mathcal{K}^*}$  is supported generically; the single comeager orbit  $\text{Aut}(\mathbf{K}) \cdot (<^*)$  has measure 1, where  $\mathbf{K}^* = \langle \mathbf{K}, <^* \rangle$ . It was asked in [AKL] whether for every uniquely ergodic Polish group  $G$  with metrizable universal minimal flow and the generic point property, the unique invariant measure on  $M(G)$  (and hence every minimal  $G$ -flow) is supported generically (the Unique Ergodicity-Generic Point Problem). We will consider the case where  $\mathcal{K}$  is the class of finite dimensional vector spaces over a finite field  $F_q$ . The companion class  $\mathcal{K}^*$  is the class of naturally ordered vector spaces, vector spaces with an ordering induced antilexicographically by an ordering of a basis and a fixed ordering of  $F_q$  with  $0 < 1$  the least elements. We

have  $\text{Flim}(\mathcal{K}) = \mathbf{V}_\infty$ , the countably infinite dimensional vector space over  $F_q$ , and we will write  $\text{Flim}(\mathcal{K}^*) = \mathbf{V}_\infty^*$ . It is shown in [AKL] that  $\text{GL}(\mathbf{V}_\infty)$  is uniquely ergodic and that the unique measure satisfies  $\mu(N_{\langle \mathbf{A}, < \rangle}) = 1/k_{\mathbf{A}}$  for each  $\mathbf{A} \subseteq \mathbf{V}_\infty$ ,  $\mathbf{A} \in \mathcal{K}$  and each  $\langle \mathbf{A}, < \rangle \in \mathcal{K}^*(\mathbf{A}) := \{\langle \mathbf{A}, < \rangle : \langle \mathbf{A}, < \rangle \in \mathcal{K}^*\}$ , where  $k_{\mathbf{A}} = |\mathcal{K}^*(\mathbf{A})|$ . We answer in the negative the above question of [AKL] by showing that  $\mu(\text{GL}(\mathbf{V}_\infty) \cdot (<^*)) = 0$ . Thus we have:

**Theorem 1.2.** *The unique invariant measure on the universal minimal flow of  $\text{GL}(\mathbf{V}_\infty)$  is not supported on the generic orbit.*

We also characterize the support of this measure and exhibit a mod zero isomorphism of  $(X_{\mathcal{K}^*}, \mu)$  with a canonical product measure.

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## 2 Amenability

Let  $(\mathcal{K}, \mathcal{K}^*)$  be an excellent pair with limits  $\mathbf{K}, \mathbf{K}^*$ . Define  $\text{Fin}(\mathbf{K})$  to be the set of all finite substructures of  $\mathbf{K}$ . Compare this notion to  $\text{Age}(\mathbf{K}) = \mathcal{K}$ ; the latter consists of all structures isomorphic to structures in  $\text{Fin}(\mathbf{K})$ . For each  $\mathbf{A} \in \text{Fin}(\mathbf{K})$ , let

$$\mathcal{K}^*(\mathbf{A}) := \{\langle \mathbf{B}, (S_1, \dots, S_n) \rangle \in \mathcal{K}^* : \mathbf{B} = \mathbf{A}\}$$

Let  $\mathbf{A}, \mathbf{B} \in \text{Fin}(\mathbf{K})$ , and let  $\pi : \mathbf{A} \rightarrow \mathbf{B}$  be an embedding. For each  $\mathbf{A}^* \in \mathcal{K}^*(\mathbf{A})$ , define:

$$\begin{aligned} \mathcal{K}^*(\mathbf{A}^*, \mathbf{B}, \pi) &:= \{\langle \mathbf{B}, (T_1, \dots, T_n) \rangle \in \mathcal{K}^*(\mathbf{B}) : \\ &\quad \pi : \langle \mathbf{A}, S_1, \dots, S_n \rangle \rightarrow \langle \mathbf{B}, T_1, \dots, T_n \rangle \text{ is an embedding}\}. \end{aligned}$$

Let  $\Omega = \bigcup_{\mathbf{A} \in \text{Fin}(\mathbf{K})} \mathcal{K}^*(\mathbf{A})$ . Though the elements of  $\Omega$  are structures, for neatness we will often denote elements of  $\Omega$  with variable names, such as  $x, y$ , etc. Form the vector space  $\mathbb{R}\Omega$ , the vector space over  $\mathbb{R}$  with basis  $\Omega$ . Let  $S$  be the set of all elements of the form:

$$x - \left( \sum_{y \in \mathcal{K}^*(x, \mathbf{B}, \pi)} y \right)$$

for some  $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ ,  $x \in \mathcal{K}^*(\mathbf{A})$ , and  $\pi : \mathbf{A} \rightarrow \mathbf{B}$  an embedding. Let  $V$  be the subspace generated by  $S$ . We are now able to state the first main theorem.

**Theorem 2.1.**  *$\text{Aut}(\mathbf{K})$  is amenable iff for all  $0 \neq v \in V$ ,  $v = \sum_{x \in \Omega} c_x x$ , there are  $x, y \in \Omega$  with  $c_x > 0$ ,  $c_y < 0$*

*Proof.*  $\Rightarrow$  Suppose  $\mu$  is an invariant measure on  $X_{\mathcal{K}^*}$ . Recall that the topology on  $X_{\mathcal{K}^*}$  is given by basic neighborhoods of the form  $N_x = \{(S_1, \dots, S_n)^* \in X_{\mathcal{K}^*} : (S_1, \dots, S_n)^*|_{\mathbf{A}} = (S_1, \dots, S_n)\}$  for  $x = \langle \mathbf{A}, (S_1, \dots, S_n) \rangle$ . Define  $\mu_\Omega : \mathbb{R}\Omega \rightarrow \mathbb{R}$  by:

$$\mu_\Omega \left( \sum_{x \in \Omega} c_x x \right) = \sum_{x \in \Omega} c_x \mu(N_x)$$

We see that for  $v \in V$ , we must have  $\mu_\Omega(v) = 0$ . As  $X_{\mathcal{K}^*}$  is minimal, we have  $\mu_\Omega(x) = \mu(N_x) > 0$  for all  $x \in \Omega$ . If  $0 \neq v \in V$ ,  $v = \sum_{x \in \Omega} c_x x$  and without loss of generality  $c_x \geq 0$  for all  $x \in \Omega$ , then for some  $x$  we have  $c_x > 0$ ; we must then have  $\mu_\Omega(v) > 0$ , a contradiction.

$\Leftarrow$  Suppose the conditions of Theorem 2.1 hold. As noted by Kechris (private communication), it is sufficient to show that for  $\mathbf{B} \in \text{Fin}(\mathbf{K})$ , there exists a consistent probability measure  $\mu_{\mathbf{B}}$  on  $\mathcal{K}^*(\mathbf{B})$ , i.e. a probability measure such that for  $\mathbf{A} \in \text{Fin}(\mathbf{K})$ ,  $x \in \mathcal{K}^*(\mathbf{A})$ , and any two embeddings  $\pi_1, \pi_2 : \mathbf{A} \rightarrow \mathbf{B}$ , we have  $\mu_{\mathbf{B}}(\mathcal{K}^*(x, \mathbf{B}, \pi_1)) = \mu_{\mathbf{B}}(\mathcal{K}^*(x, \mathbf{B}, \pi_2))$ . Indeed, if this is the case, let  $\mathbf{A}_1 \subset \mathbf{A}_2 \subset \dots \subset \mathbf{K}$  be finite substructures with  $\bigcup_{n=1}^\infty \mathbf{A}_n = \mathbf{K}$ , and for each  $n$  let  $\mu_n$  be a consistent probability measure on  $\mathcal{K}^*(\mathbf{A}_n)$ . We will create an invariant measure on  $X_{\mathcal{K}^*}$  as follows: let  $\mathbf{A} \in \text{Fin}(\mathbf{K})$  and  $x \in \mathcal{K}^*(\mathbf{A})$ ; note that  $\mathbf{A} \subset \mathbf{A}_n$  for large enough  $n$ . Set  $\mu(N_x) = \lim_{n \rightarrow \mathcal{U}} \mu_n(\mathcal{K}^*(x, \mathbf{A}_n, i_{\mathbf{A}}))$ , where  $\mathcal{U}$  is a non-principal ultrafilter on  $\mathbb{N}$  and  $i_{\mathbf{A}}$  is the inclusion embedding. To see that this is invariant, let  $\mathbf{B} \in \text{Fin}(\mathbf{K})$  and let  $\pi : \mathbf{A} \rightarrow \mathbf{B}$  be an embedding. Let  $n$  be sufficiently large so that  $\mathbf{A}, \mathbf{B} \subset \mathbf{A}_n$ . Now we have  $\mu_n(\mathcal{K}^*(x, \mathbf{A}_n, i_{\mathbf{A}})) = \mu_n(\mathcal{K}^*(x, \mathbf{A}_n, i_{\mathbf{B}} \circ \pi))$ , and  $\mathcal{K}^*(x, \mathbf{A}_n, i_{\mathbf{B}} \circ \pi) = \bigsqcup_{y \in \mathcal{K}^*(x, \mathbf{B}, \pi)} \mathcal{K}^*(y, \mathbf{A}_n, i_{\mathbf{B}})$ .

Let  $S_{\mathbf{B}} \subset V$  consist of all elements of  $\mathbb{R}\Omega$  of the form:

$$\left( \sum_{y \in \mathcal{K}^*(x, \mathbf{B}, \pi_1)} y \right) - \left( \sum_{z \in \mathcal{K}^*(x, \mathbf{B}, \pi_2)} z \right) = \left( x - \left( \sum_{z \in \mathcal{K}^*(x, \mathbf{B}, \pi_2)} z \right) \right) - \left( x - \left( \sum_{y \in \mathcal{K}^*(x, \mathbf{B}, \pi_1)} y \right) \right)$$

for some  $\mathbf{A} \in \text{Fin}(\mathbf{K})$ ,  $x \in \mathcal{K}^*(\mathbf{A})$ , and embeddings  $\pi_1, \pi_2 : \mathbf{A} \rightarrow \mathbf{B}$ . Consider the following system of inequalities and equalities in real variables  $q_x$ ,  $x \in \mathcal{K}^*(\mathbf{B})$ , where for  $v = \sum_{x \in \Omega} c_x x \in \mathbb{R}\Omega$ , we let  $q_v = \sum_{x \in \Omega} c_x q_x$ :

$$q_s = 0 \quad (s \in S_{\mathbf{B}}) \tag{1}$$

$$q_x > 0 \quad (x \in \mathcal{K}^*(\mathbf{B})) \tag{2}$$

$$\sum_{x \in \mathcal{K}^*(\mathbf{B})} q_x = 1 \tag{3}$$

If this system has a solution, the solution is the consistent probability measure we seek, with  $\mu_{\mathbf{B}}(x) = q_x$ . Note that the system (1), (2), (3) has a solution iff the system (1), (2) has a solution. Now we can use Stiemke's theorem ([S], see also Border [B]):

**Theorem 2.2** (Stiemke). *For  $A$  a real-valued matrix, the equation  $A\mathbf{x} = 0$  has a solution with  $\mathbf{x} > 0$  (each entry  $x \in \mathbf{x}$  has  $x > 0$ ), or the equation  $\mathbf{y}^T A \gneq 0$  has a solution (each entry  $z \in \mathbf{y}^T A$  has  $z \geq 0$ , and at least one entry has  $z > 0$ ).*

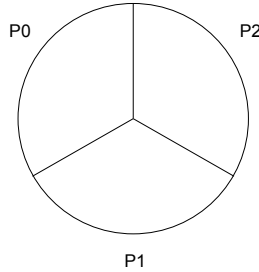
Suppose the system (1), (2) doesn't have a solution. To use Stiemke's theorem, let  $A$  be the matrix of coefficients from (1). Now the entries  $y_s$  of  $\mathbf{y}$  correspond to scalar multiples of each equation  $q_s = 0$ . Consider the element  $v = \sum_{s \in S_{\mathbf{B}}} y_s s \in V$ . Write

$v = \sum_{x \in \mathcal{K}^*(\mathbf{B})} c_x x$ . Stiemke's theorem tells us that  $c_x \geq 0$  for all  $x \in \mathcal{K}^*(\mathbf{B})$  and that for some  $x$  we have  $c_x > 0$ . This contradicts the assumptions of Theorem 2.1.

□

### 3 Application: $\text{Aut}(\mathbf{S}(3))$ is not amenable

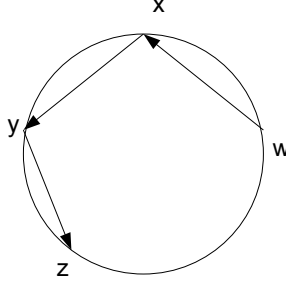
$\mathbf{S}(3)$  is the directed graph with vertex set  $\{e^{iq} : q \in \mathbb{Q}\}$ . For vertices  $a = e^{iq_1}$ ,  $b = e^{iq_2}$ , set  $a \rightarrow b$  iff there is  $m \in \mathbb{Z}$  with  $q_2 - q_1 \in (2\pi m, 2\pi m + 2\pi/3)$ , i.e.  $b$  is less than  $120^\circ$  counterclockwise from  $a$ . We will write  $(a, b)$  for those points lying in the interval counterclockwise from  $a$  to  $b$ . Let  $\mathcal{K} = \text{Age}(\mathbf{S}(3))$ . The appropriate companion class with the Expansion and Ramsey properties is  $\mathcal{K}^* = \text{Age}(\mathbf{S}(3)^*)$ , where  $\mathbf{S}(3)^* = \langle \mathbf{S}(3), P_0, P_1, P_2 \rangle$ , and the  $P_i$  are unary relations with  $P_i(a)$  iff  $a$  is between  $120i$  and  $120(i+1)$  degrees counterclockwise from the top of the circle (see [LNV T]). For  $a \in \mathbf{A}^* \in \mathcal{K}^*$ , we will write  $P(a) = i$  when  $P_i(a)$  holds.



We will use Theorem 2.1 to show:

**Theorem 3.1.**  $\text{Aut}(\mathbf{S}(3))$  is not amenable.

*Proof.* Let  $\mathbf{A} \in \text{Fin}(\mathbf{K})$  be a directed graph consisting of two vertices  $a$  and  $b$  with no edge between them. Let  $\mathbf{B} \in \text{Fin}(\mathbf{K})$  be a directed graph with vertices  $w, x, y$ , and  $z$ , with edges  $w \rightarrow x$ ,  $x \rightarrow y$ , and  $y \rightarrow z$ .



Consider  $\mathcal{K}^*(\mathbf{B})$ ; we list all 12 expansions of  $\mathbf{B}$  below, where  $(i, j, k, l)$  stands for  $\langle \mathbf{B}, (P_1, P_2, P_3) \rangle \in \mathcal{K}^*(\mathbf{B})$  with  $P_i(w), P_j(x), P_k(y), P_l(z)$ :

$$\begin{aligned} \mathcal{K}^*(\mathbf{B}) = \{ & (0, 1, 1, 2), (0, 1, 2, 2), (1, 1, 2, 2), (1, 1, 2, 0), \\ & (1, 2, 2, 0), (1, 2, 0, 0), (2, 2, 0, 0), (2, 2, 0, 1), \\ & (2, 0, 0, 1), (2, 0, 1, 1), (0, 0, 1, 1), (0, 0, 1, 2) \} \end{aligned}$$

To arrive at the above, first observe that we cannot have  $P(w) = P(x) = P(y)$  nor  $P(x) = P(y) = P(z)$ , as this would require  $w \rightarrow y$  or  $x \rightarrow z$ , respectively. Also observe that mod 3, we cannot have any of  $P(w) = P(x) + 1$ ,  $P(x) = P(y) + 1$ , nor  $P(y) = P(z) + 1$ ; this is because for any  $a, b \in \mathbf{S}(3)^*$ , we have  $a \rightarrow b \Rightarrow P(a) \neq P(b) + 1$ . Lastly, we may not have  $P(w) = P(z)$ , as this would imply  $z \rightarrow w$  or  $w \rightarrow z$ . Now without loss of generality fix  $w \in P_1$ . There are exactly four possible expansions meeting the necessary conditions, and all four are easily realized. The other eight expansions are then given by adding 1 or 2 mod 3 to each coordinate

Let  $x \in \mathcal{K}^*(\mathbf{A})$  denote the expansion with  $P_0(a), P_1(b)$ . Let  $\pi_1 : \mathbf{A} \rightarrow \mathbf{B}$  denote the embedding  $\pi_1(a) = w, \pi_1(b) = y$ . Let  $\pi_2 : \mathbf{A} \rightarrow \mathbf{B}$  denote the embedding  $\pi_2(a) = w, \pi_2(b) = z$ . Now we have:

$$\begin{aligned} \mathcal{K}^*(x, \mathbf{B}, \pi_1) &= \{(0, 1, 1, 2), (0, 0, 1, 1), (0, 0, 1, 2)\} \\ \mathcal{K}^*(x, \mathbf{B}, \pi_2) &= \{(0, 0, 1, 1)\} \end{aligned}$$

Forming the spaces  $\mathbb{R}\Omega, V$  as above, we have that:

$$\begin{aligned} [(0, 1, 1, 2) + (0, 0, 1, 1) + (0, 0, 1, 2)] - [(0, 0, 1, 1)] &\in V \\ \Rightarrow (0, 1, 1, 2) + (0, 0, 1, 2) &\in V \end{aligned}$$

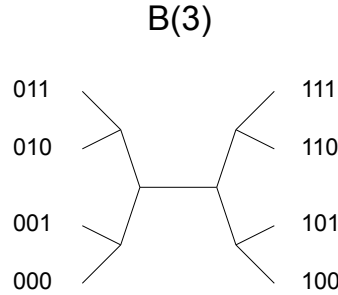
Hence  $\text{Aut}(\mathbf{S}(3))$  is not amenable. □

## 4 Application: Boron Trees

A boron tree is a graph-theoretic unrooted tree where each vertex has degree 1 or 3. It is possible to interpret boron trees as a model theoretic structure such that the class of finite boron trees forms a Fraïssé class; we will closely follow the exposition of Jasiński

[J]. For a boron tree  $T$ , let  $(T) = \langle L_T, R \rangle$  be the structure with universe  $L_T$ , the leaves of  $T$ , and  $R$  a 4-ary relation defined as follows: for leaves  $a, b, c, d$ , we have  $R(a, b, c, d)$  if  $a, b, c, d$  are distinct and there are paths from  $a$  to  $b$  and from  $c$  to  $d$  which do not intersect. It is a fact that for finite boron trees  $T$  and  $U$ ,  $T \cong U$  iff  $(T) \cong (U)$ .

Let  $\mathbf{B}(n)$  be the boron tree structure defined as follows: its universe is  $B_n = 2^n = \{f : n \rightarrow 2\}$ . Now for any two leaves  $a$  and  $b$ , let  $\delta(a, b) = \max(k : a|_k = b|_k)$ . Define  $[a, b] = \{a|_k : k \in [\delta(a, b), n]\} \cup \{b|_k : k \in [\delta(a, b), n]\}$ . Now for distinct  $a, b, c, d$ , set  $R(a, b, c, d)$  if  $[a, b] \cap [c, d] = \emptyset$ . The associated tree has vertices  $2^{\leq n} \setminus \emptyset$ ; vertices  $u : k \rightarrow 2$  and  $v : k+1 \rightarrow 2$  are adjacent if  $v|_k = u$ , and additionally the vertices  $x_0, x_1 : 1 \rightarrow 2$  with  $x_0(0) = 0, x_1(0) = 1$  are adjacent.



On each  $\mathbf{B}(n)$ , let  $<_l$  be the lexicographic linear ordering of the leaves;  $a <_l b$  if  $a(0) < b(0)$  or  $a(0) = b(0)$  and  $a(1) < b(1)$  or etc. Also, for any function  $f : k \rightarrow 2$ ,  $k \leq n$ , let  $B_f(n)$  consist of those  $a \in B(n)$  with  $a|_k = f$ .

Let  $\mathcal{B}$  be the class of finite boron tree structures. It is a fact that the set  $\{\mathbf{B}(n) : n \in \mathbb{N}\}$  is cofinal in  $\mathcal{B}$ ; for each  $\mathbf{A} \in \mathcal{B}$ , there is an  $n$  for which there is an embedding  $\pi : \mathbf{A} \rightarrow \mathbf{B}(n)$ . We will use this fact in defining the companion class  $\mathcal{B}^*$  such that  $(\mathcal{B}, \mathcal{B}^*)$  is an excellent pair. Let  $\mathbf{A} \in \mathcal{B}$ , and let  $\pi : \mathbf{A} \rightarrow \mathbf{B}(n)$  be an embedding for some  $n$ . Define the structure  $o(\mathbf{A}, \pi) = \langle \mathbf{A}, S \rangle$  as follows:  $S$  is a 3-ary relation, where for  $a, b, c \in A$ , we have  $S(a, b, c)$  if  $\pi(a), \pi(b) <_l \pi(c)$  and  $\delta(\pi(a), \pi(b)) > \delta(\pi(b), \pi(c))$ . We can rephrase this condition in a somewhat easier to grasp way: for  $x, y \in B(n)$ , let  $M(x, y) \in B(n)$  be as follows:

$$M(x, y)(j) = \begin{cases} x(j) & \text{if } j < \delta(x, y) \\ 1 & \text{if } j \geq \delta(x, y) \end{cases}$$

Now we have  $S(a, b, c)$  iff  $M(\pi(a), \pi(b)) <_l \pi(c)$ . Note that  $S(a, b, c)$  iff  $S(b, a, c)$ . Let  $\mathcal{B}^*$  be the class of all such  $o(\mathbf{A}, \pi)$ . It is a fact that  $(\mathcal{B}, \mathcal{B}^*)$  is an excellent pair (see [J]), with limits  $\mathbf{T}, \mathbf{T}^*$ . We will show:

**Theorem 4.1.**  *$\text{Aut}(\mathbf{T})$  is not amenable.*

*Proof.* We begin with the following proposition.

**Proposition 4.2.** *Let  $o(\mathbf{A}, \pi) \in \mathcal{B}^*$ , with  $\pi : \mathbf{A} \rightarrow \mathbf{B}(n)$ , and  $|A| = l$ . Then there exists an embedding  $\phi : \mathbf{A} \rightarrow \mathbf{B}(l-1)$  with  $o(\mathbf{A}, \pi) = o(\mathbf{A}, \phi)$ .*

*Proof.* Suppose there exists  $k \in \{0, 1, \dots, n-1\}$  such that for any  $a, b \in A$ , we have  $\pi(a)|_k = \pi(b)|_k \Rightarrow \pi(a)|_{k+1} = \pi(b)|_{k+1}$ , or equivalently that for no  $a, b \in A$  do we have  $\delta(\pi(a), \pi(b)) = k$ . We will show that there is an embedding  $\psi : \mathbf{A} \rightarrow \mathbf{B}(n-1)$  with  $o(\mathbf{A}, \psi) = o(\mathbf{A}, \pi)$ . For  $a \in B(n)$ , define the map  $f_k : B(n) \rightarrow B(n-1)$  as follows:

$$f_k(a)(j) = \begin{cases} a(j) & \text{if } j < k \\ a(j+1) & \text{if } j \geq k \end{cases}$$

Now define  $\psi : A \rightarrow B(n-1)$  by  $\psi = f_k \circ \pi$ . First let us show that  $\psi$  is an embedding  $\mathbf{A} \rightarrow \mathbf{B}(n-1)$ . It will be useful to note that we have:

$$\delta(\psi(a), \psi(b)) = \begin{cases} \delta(\pi(a), \pi(b)) & \text{if } \delta(\pi(a), \pi(b)) < k \\ \delta(\pi(a), \pi(b)) - 1 & \text{if } \delta(\pi(a), \pi(b)) > k \end{cases}$$

Let  $a, b, c, d \in A$  be distinct, and suppose  $R(a, b, c, d)$ . As  $\pi$  is an embedding, we have  $[\pi(a), \pi(b)] \cap [\pi(c), \pi(d)] = \emptyset$ . Suppose, without loss of generality and for sake of contradiction, that  $\psi(a)|_m = \psi(c)|_m \in [\psi(a), \psi(b)] \cap [\psi(c), \psi(d)]$ . If  $m < k$ , we have that  $\pi(a)|_m = \pi(c)|_m \in [\pi(a), \pi(b)] \cap [\pi(c), \pi(d)]$ , a contradiction. If  $m \geq k$ , then  $\pi(a)|_{m+1} \in [\pi(a), \pi(b)]$  and  $\pi(c)|_{m+1} \in [\pi(c), \pi(d)]$ . Therefore we must have  $\pi(a)|_{m+1} \neq \pi(c)|_{m+1}$ . However, this is only possible if  $\pi(a)(k) \neq \pi(c)(k)$ . As  $\pi(a)|_k = \pi(c)|_k$ , this is a contradiction. Therefore we have  $R(\psi(a), \psi(b), \psi(c), \psi(d))$ .

If  $\neg R(a, b, c, d)$ , we may suppose  $\pi(a)|_m = \pi(c)|_m \in [\pi(a), \pi(b)] \cap [\pi(c), \pi(d)]$ . If  $m < k$ , then  $\psi(a)|_m = \psi(c)|_m \in [\psi(a), \psi(b)] \cap [\psi(c), \psi(d)]$ . If  $m \geq k$ , then  $\psi(a)|_{m-1} = \psi(c)|_{m-1} \in [\psi(a), \psi(b)] \cap [\psi(c), \psi(d)]$ . Hence we have  $\neg R(\psi(a), \psi(b), \psi(c), \psi(d))$ .

Now we will show that  $o(\mathbf{A}, \psi) = o(\mathbf{A}, \pi)$ . Suppose  $S^\pi(a, b, c)$ , or equivalently  $M(\pi(a), \pi(b)) <_l \pi(c)$ . Observe that  $M(\psi(a), \psi(b)) = f_k(M(\pi(a), \pi(b)))$ . Now if  $\psi(c) <_l M(\psi(a), \psi(b))$ , we must have  $\pi(c)|_k = M(\pi(a), \pi(b))|_k$ ; it follows that  $\delta(\pi(a), \pi(b)) > k$ . Let  $m$  be least such that  $\pi(c)(m) \neq \pi(a)(m)$ . Then we must have  $k < m < \delta(\pi(a), \pi(b))$ . Then  $m-1$  is least such that  $\psi(c) \neq \psi(a)$ , and  $m-1 < \delta(\psi(a), \psi(c))$ . Hence  $S^\psi(a, b, c)$ .

Now suppose  $S^\psi(a, b, c)$ . Let  $m$  be least such that  $\psi(c)(m) \neq \psi(a)(m)$ . Then  $m < \delta(\psi(a), \psi(b)) \leq \delta(\pi(a), \pi(b))$ . Now if  $m < k$ , then  $m$  is least such that  $\pi(c)(m) \neq \pi(a)(m)$ . If  $m > k$ , then  $m+1$  is least such that  $\pi(c)(m+1) \neq \pi(a)(m+1)$ , and  $m+1 < \delta(\pi(a), \pi(b))$ . Hence  $S^\pi(a, b, c)$ , and  $o(\mathbf{A}, \psi) = o(\mathbf{A}, \pi)$ .

Now suppose no such  $k$  exists. We will show that  $|A| \geq n+1$ . More precisely, we will show that for any  $m$  and embedding  $\phi : \mathbf{A} \rightarrow \mathbf{B}(m)$ , the map  $\delta_\phi : A \times A \rightarrow m+1$  with  $\delta_\phi(a, b) = \delta(\phi(a), \phi(b))$  has  $|\delta_\phi(A \times A)| \leq n$ . For  $n=1$  this is clear. Assume the result true for  $n=l$ , and suppose  $|A| = l+1$ . Fix  $a \in A$ , and let  $b \neq a$  be such that  $\delta_\phi(a, b)$  is maximal. Let  $c \in A$  with  $c \neq a$  and  $c \neq b$ . Then we have:

$$\delta_\phi(a, c) = \begin{cases} \delta_\phi(a, b) & \text{if } \delta_\phi(b, c) > \delta_\phi(a, b) \\ \delta_\phi(b, c) & \text{if } \delta_\phi(b, c) < \delta_\phi(a, b) \end{cases}$$

We see that  $|\delta_\phi(A \times A)| = |\delta_\phi(A \setminus a \times A \setminus a) \cup \{\delta_\phi(a, b)\}| \leq l+1$  □

With this result, we can now compute  $\mathcal{B}^*(\mathbf{A})$  for  $\mathbf{A} \in \text{Fin}(\mathbf{T})$ ; we will exhibit an element of  $V \subset \mathbb{R}\Omega$  showing that  $\text{Aut}(\mathbf{T})$  is not amenable. Let  $\mathbf{A} \in \text{Fin}(\mathbf{T})$  be a boron tree with three leaves  $a, b, c$ . We list the 12 expansions in  $\mathcal{B}^*(\mathbf{A})$  below; we will denote an element of  $\mathcal{B}^*(\mathbf{A})$  by  $[a_1 < \dots < a_k; (x_1, y_1, z_1), \dots, (x_l, y_l, z_l)]$ , where we have  $S(a_i, a_i, a_j)$  for  $i < j$ ,  $S(x_i, y_i, z_i)$  and  $S(y_i, x_i, z_i)$  for  $1 \leq i \leq l$ :

$$\begin{aligned} \mathcal{B}^*(\mathbf{A}) = \{ & [a < b < c; \emptyset], [a < c < b; \emptyset], [b < a < c; \emptyset], \\ & [b < c < a; \emptyset], [c < a < b; \emptyset], [c < b < a; \emptyset], \\ & [a < b < c; (a, b, c)], [a < c < b; (a, c, b)], [b < a < c; (b, a, c)], \\ & [b < c < a; (b, c, a)], [c < a < b; (c, a, b)], [c < b < a; (c, b, a)] \} \end{aligned}$$

Now consider  $\mathbf{B}(2) \in \text{Fin}(\mathbf{T})$ , where  $B(2) = \{w = 00, x = 01, y = 10, z = 11\}$ ; this structure has 40 expansions. These can be split into 5 types, each of which has 8 expansions.

1. Type *A*: Expansions of the form  $o(\mathbf{B}(2), \pi)$  with  $\pi : \mathbf{B}(2) \rightarrow \mathbf{B}(2)$ . Below,  $(s, t, u, v)$  stands for the expansion  $[s < t < u < v; (s, t, u), (s, t, v)]$ :

$$\begin{aligned} A = \{a_1, \dots, a_8\} = \{ & (w, x, y, z), (w, x, z, y), (x, w, y, z), (x, w, z, y), \\ & (y, z, w, x), (y, z, x, w), (z, y, w, x), (z, y, x, w) \} \end{aligned}$$

2. Type *B*: Expansions of the form  $o(\mathbf{B}(2), \pi)$  with  $\pi : \mathbf{B}(2) \rightarrow \mathbf{B}(3)$  such that  $|B_0(3) \cap \pi(A)| = 1$  and  $|B_{10}(3) \cap \pi(A)| = 1$ . Below,  $(s, t, u, v)$  stands for the expansion  $[s < t < u < v; \emptyset]$ :

$$\begin{aligned} B = \{b_1, \dots, b_8\} = \{ & (w, x, y, z), (w, x, z, y), (x, w, y, z), (x, w, z, y), \\ & (y, z, w, x), (y, z, x, w), (z, y, w, x), (z, y, x, w) \} \end{aligned}$$

3. Type *C*: Expansions of the form  $o(\mathbf{B}(2), \pi)$  with  $\pi : \mathbf{B}(2) \rightarrow \mathbf{B}(3)$  such that  $|B_0(3) \cap \pi(A)| = 1$  and  $|B_{11}(3) \cap \pi(A)| = 1$ . Below,  $(s, t, u, v)$  stands for the expansion  $[s < t < u < v; (t, u, v)]$ :

$$\begin{aligned} C = \{c_1, \dots, c_8\} = \{ & (w, y, z, x), (w, z, y, x), (x, y, z, w), (x, z, y, w), \\ & (y, w, x, z), (y, x, w, z), (z, w, x, y), (z, x, w, y) \} \end{aligned}$$

4. Type *D*: Expansions of the form  $o(\mathbf{B}(2), \pi)$  with  $\pi : \mathbf{B}(2) \rightarrow \mathbf{B}(3)$  such that  $|B_1(3) \cap \pi(A)| = 1$  and  $|B_{00}(3) \cap \pi(A)| = 1$ . Below,  $(s, t, u, v)$  stands for the expansion  $[s < t < u < v; (s, t, v), (s, u, v), (t, u, v)]$ :

$$\begin{aligned} D = \{d_1, \dots, d_8\} = \{ & (w, y, z, x), (w, z, y, x), (x, y, z, w), (x, z, y, w), \\ & (y, w, x, z), (y, x, w, z), (z, w, x, y), (z, x, w, y) \} \end{aligned}$$

5. Type *E*: Expansions of the form  $o(\mathbf{B}(2), \pi)$  with  $\pi : \mathbf{B}(2) \rightarrow \mathbf{B}(3)$  such that  $|B_1(3) \cap \pi(A)| = 1$  and  $|B_{01}(3) \cap \pi(A)| = 1$ . Below,  $(s, t, u, v)$  stands for the expansion  $[s < t < u < v; (s, t, u), (s, t, v), (s, u, v), (t, u, v)]$ :

$$E = \{e_1, \dots, e_8\} = \{(w, x, y, z), (w, x, z, y), (x, w, y, z), (x, w, z, y), \\ (y, z, w, x), (y, z, x, w), (z, y, w, x), (z, y, x, w)\}$$

Now set  $x = [a < b < c; (a, b, c)] \in \mathcal{B}^*(\mathbf{A})$ . Let  $\pi_1, \pi_2 : \mathbf{A} \rightarrow \mathbf{B}(2)$  with  $\pi_1(a, b, c) = (w, x, y)$ ,  $\pi_2(a, b, c) = (w, y, z)$ . We see that  $\mathcal{B}^*(x, \mathbf{B}(2), \pi_1) = \{a_1, a_2, c_7, d_7, e_1, e_2\}$  and  $\mathcal{B}^*(x, \mathbf{B}(2), \pi_2) = \{e_1, e_3\}$ . Thus in  $\mathbb{R}\Omega$ :

$$a_1 + a_2 + c_7 + d_7 + e_2 - e_3 \in V$$

Now let  $\phi_1, \phi_2 : \mathbf{B}(2) \rightarrow \mathbf{B}(2)$  with  $\phi_1(w, x, y, z) = (w, x, y, z)$ ,  $\phi_2(w, x, y, z) = (x, w, z, y)$ . We have that  $\mathcal{B}^*(e_2, \mathbf{B}(2), \phi_1) = \{e_2\}$ , and  $\mathcal{B}^*(e_2, \mathbf{B}(2), \phi_2) = \{e_3\}$ . In  $\mathbb{R}\Omega$ :

$$e_2 - e_3 \in V \\ \Rightarrow a_1 + a_2 + c_7 + d_7 \in V$$

Therefore  $\text{Aut}(\mathbf{T})$  is not amenable. □

## 5 Proof of Theorem 1.2

For the remainder of this paper, we shift our focus to the classes  $\mathcal{K}$ , the class of finite dimensional vector spaces over  $F_q$ , and  $\mathcal{K}^*$ , the class of such vector spaces equipped with a natural linear ordering, an ordering induced antilexicographically by some ordered basis and a fixed ordering of  $F_q$  with  $0 < 1$  the least elements. In what follows, all vector spaces are assumed to be over  $F_q$ . We remind the reader that we have  $\text{Flim}(\mathcal{K}) = \mathbf{V}_\infty$ , the countably infinite dimensional vector space over  $F_q$ , and  $\text{Flim}(\mathcal{K}^*) = \mathbf{V}_\infty^*$ .

Given any vector space  $V$  with any ordering  $<$ , say that  $\langle V, < \rangle$  has the *Finite Lex Property* (FLP) if for any finite dimensional subspace  $U \subset V$ , we have that  $<|_U$  is a natural ordering. In particular, a finite dimensional, ordered vector space has the FLP iff that ordering is natural (See Thomas [Th]), and  $\langle \mathbf{V}_\infty, < \rangle$  has the FLP iff  $< \in X_{\mathcal{K}^*}$ . Given  $v \in V$ ,  $v \neq 0$ , say that  $v$  is *minimal in its line* if  $v \leq cv$  for all  $c \in F_q \setminus \{0\}$ . Given  $u, v \in V$ , define the relation  $u \ll v \Leftrightarrow cu < dv$  for all  $c, d \in F_q \setminus \{0\}$ . Also define  $u \sim v \Leftrightarrow (u \not\ll v) \wedge (v \not\ll u)$ . For  $q = 2$ , we need to tweak these definitions a bit. We define  $u \sim v$  to hold when  $u + v < \min(u, v)$ . We have  $u \ll v$  if  $u < v$  and  $u \approx v$ . Note that if  $u \sim v$ , then  $cu \sim dv$  for all  $c, d \in F_q \setminus \{0\}$ . Also, if  $u, v \in \langle V, <|_V \rangle \subset \langle U, < \rangle$ , then  $u \ll_V v \Leftrightarrow u \ll_U v$  and  $u \sim_V v \Leftrightarrow u \sim_U v$ .

**Lemma 5.1.** *For  $\langle V, < \rangle$  with the FLP,  $\sim$  is an equivalence relation.*

*Proof.* Suppose  $u \sim v$  and  $v \sim w$ . Form  $\langle u, v, w \rangle := U$ , and let  $\{x_0 > \dots > x_l\}$  be the basis of  $U$  inducing  $<|_U$ ,  $l \leq 2$ . Write  $u = u_0x_0 + \dots + u_lx_l$ ,  $u_i \in F_q$ ; likewise for  $v, w$ . Observe that one of  $u_0, v_0, w_0$  is nonzero; as  $u \sim v$  and  $v \sim w$ , all three must be nonzero. It follows that  $u \sim w$ . □

When  $\langle V, < \rangle$  is finite dimensional,  $\sim$  has a simple characterization. Say  $<$  is given by ordered basis  $Z = \{z_0 > \dots > z_{n-1}\}$ . For  $v \in V \setminus \{0\}$ , write  $v = v_0 z_0 + \dots + v_{n-1} z_{n-1}$ , and suppose  $l$  is least with  $v_l$  nonzero. Then for  $u = u_0 z_0 + \dots + u_{n-1} z_{n-1}$ , we have  $u \sim v$  iff  $l$  is also least with  $u_l$  nonzero.

Observe that  $V/\sim$ , the set of nonzero equivalence classes, is also linearly ordered by  $A < B \Leftrightarrow a < b$  for all  $a \in A, b \in B$ . We will denote by  $</\sim$  the order type of this linear ordering. It will be useful to introduce a standard notation for the  $n$  nonzero equivalence classes of an  $n$ -dimensional, naturally ordered vector space  $\langle V, < \rangle$ . Call these classes  $[V, <]_{n-1} < \dots < [V, <]_0$ ; note that  $v \in [V, <]_l$  iff  $l$  is least with  $v_l$  nonzero.

Let  $V_1 \subset W$  be a 1-dimensional subspace. For  $\langle W, < \rangle$  with the FLP, observe that for any  $u, v \in V_1 \setminus \{0\}$  we have  $u \sim v$ . Call this equivalence class  $[V_1, <]$ . Define:

$$N_{V_1}^k = \{< \in X_{\mathcal{K}^*} : [V_1, <] < [u] \text{ for at most } k \text{ equivalence classes } [u] \in \mathbf{V}_\infty / \sim\}$$

Note that  $N_{V_1}^k \subseteq N_{V_1}^l$  when  $k \leq l$ . Let  $N_{V_1}^{fin} = \bigcup_{k \in \mathbb{N}} N_{V_1}^k$ . Notice that for any  $< \in N_{V_1}^{fin}$ , the extension property does not hold for  $\langle \mathbf{V}_\infty, < \rangle$ . To see this, let  $\langle V, <|_V \rangle \subset \langle U, < \rangle \in \mathcal{K}^*$ , where  $U$  is  $k+2$ -dimensional,  $<$  on  $U$  is given by an ordered basis  $u_0 > \dots > u_{k+1}$ , and  $V = \langle u_{k+1} \rangle$ . Now given  $< \in X_{\mathcal{K}^*}$ , let  $\pi : \langle V, <|_V \rangle \rightarrow \langle \mathbf{V}_\infty, < \rangle$  with  $\pi(V) = V_1$ . There are at most  $k$  equivalence classes greater than  $[V_1, <]$  in  $\mathbf{V}_\infty$ , but  $k+1$  equivalence classes greater than  $[V, <] = [U, <]_{k+1}$  in  $U$ . It follows that there is no  $\pi' : \langle U, < \rangle \rightarrow \langle \mathbf{V}_\infty, < \rangle$  extending  $\pi$ .

Conversely, if there are at least  $k+1$  equivalence classes above  $[V_1, <]$  in  $\langle \mathbf{V}_\infty, < \rangle$ , then  $\pi$  admits an extension  $\pi'$ . We can see this as follows: choose representatives  $w_i$ ,  $0 \leq i \leq k$  from  $k+1$  equivalence classes  $[w_0] > \dots > [w_k]$ , where  $[w_k] > [V_1, <]$ . Let  $W = \langle w_0, \dots, w_k, V_1 \rangle \subset \mathbf{V}_\infty$ . Then  $W$  is a  $k+2$  dimensional space; let  $\{x_0 > \dots > x_{k+1}\}$  be the basis which gives the ordering  $<|_W$ . Then we see that  $\langle x_{k+1} \rangle = V_1$ , and an extension  $\pi'$  is given by letting  $\pi'(u_i) = x_i$ .

Let  $W \subset \mathbf{V}_\infty$  be a finite dimensional subspace with  $V_1 \subset W$ , and let  $< \in X_{\mathcal{K}^*}$ . Define  $(N_W)_{V_1}^k$  to be those  $< \in X_{\mathcal{K}^*}$  for which  $[V_1, <|_W] \geq [W, <|_W]_k$ , i.e. those orderings for which there are at most  $k$  equivalence classes of  $W/\sim$  greater than  $[V_1, <|_W]$ ; if  $V_1 \not\subset W$ , set  $(N_W)_{V_1}^k = X_{\mathcal{K}^*}$ . Note that  $(N_U)_{V_1}^k \subseteq (N_W)_{V_1}^k$  for  $W \subseteq U$ . Hence  $N_{V_1}^k = \bigcap_{W \in \text{Fin}(\mathbf{V}_\infty)} (N_W)_{V_1}^k$ ,

where  $\text{Fin}(\mathbf{K})$  denotes the set of finite substructures of  $\mathbf{K}$ . If  $V_1 = W_1 \subset W_2 \subset \dots$  where each  $W_n$  is  $n$ -dimensional and  $\mathbf{V}_\infty = \bigcup_{n=1}^\infty W_n$ , we have  $N_{V_1}^k = \bigcap_{n=1}^\infty (N_{W_n})_{V_1}^k$ . Whether or not  $< \in (N_W)_{V_1}^k$  depends only on  $<|_W$ ; each  $(N_W)_{V_1}^k$  is a union of basic open sets of the form  $N_{\langle W, <|_W \rangle} := \{< \in X_{\mathcal{K}^*} : <|_W = <|_W\}$ .

Let us compute  $\mu((N_W)_{V_1}^k)$ . We have:

$$\mu((N_W)_{V_1}^k) = \frac{\#(<|_W \text{ on } W \text{ with } [V_1, <|_W] \geq [W, <|_W]_k)}{\#(<|_W \text{ on } W)}$$

Fix  $v \in V_1 \setminus \{0\}$ . For  $(v_0, \dots, v_{m-1}) \in F_q$  not all zero, let

$$N_{(v_0, \dots, v_{m-1})} = \{<|_W : <|_W \text{ is given by ordered basis } b_0 >_W \dots >_W b_{m-1} \\ \text{and } v = v_0 b_0 + \dots + v_{m-1} b_{m-1}\}$$

First let us show that for any two nonzero  $(u_i)_{i=0}^{m-1}$ ,  $(v_i)_{i=0}^{m-1}$ , we have  $|N_{(u_0, \dots, u_{m-1})}| = |N_{(v_0, \dots, v_{m-1})}|$ . Let  $l$  be least with  $v_l$  nonzero. Select  $b_0, \dots, b_{l-1}, b_{l+1}, \dots, b_{m-1}$  arbitrarily such that they are linearly independent and don't span  $V_1$ . There are  $(q^m - q)(q^m - q^2) \cdots (q^m - q^{m-1})$  ways to do this. Now set:

$$b_l = v_l^{-1}(v - v_0 b_0 - \dots - v_{l-1} b_{l-1} - v_{l+1} b_{l+1} - \dots - v_{m-1} b_{m-1})$$

This ordered basis gives some  $<_W \in N_{(v_0, \dots, v_{m-1})}$ , and certainly each  $<_W$  can be uniquely produced in this manner. Hence  $|N_{(v_0, \dots, v_{m-1})}| = (q^m - q)(q^m - q^2) \cdots (q^m - q^{m-1})$ . Now we have:

$$\begin{aligned} \mu((N_W)_{V_1}^k) &= \frac{\sum \{|N_{(v_0, \dots, v_{m-1})}| : (v_i)_{i=0}^{m-1} \neq 0 \text{ and there is } l \leq k \text{ with } v_l \neq 0\}}{\sum \{|N_{(v_0, \dots, v_{m-1})}| : (v_i)_{i=0}^{m-1} \neq 0\}} \\ &= \frac{q^m - q^{m-k-1}}{q^m - 1} \\ &= q^{m-k-1} \left( \frac{q^{k+1} - 1}{q^m - 1} \right) \end{aligned}$$

Now we have  $\mu(N_{V_1}^k) = \lim_{m \rightarrow \infty} q^{m-k-1} \left( \frac{q^{k+1} - 1}{q^m - 1} \right) = 1 - \frac{1}{q^{k+1}}$ . Letting  $k \rightarrow \infty$ , we have  $\mu(N_{V_1}^{fin}) = 1$ . This proves Theorem 1.1. As there are countably many 1-dimensional subspaces of  $\mathbf{V}_\infty$ , we obtain the following immediate corollary.

**Corollary 5.2.** *Let  $N_{\omega^*} = \{< \in X_{\mathcal{K}^*} : < / \sim = \omega^*\}$ . Then  $\mu(N_{\omega^*}) = 1$ .*

*Proof.* We will show that  $N_{\omega^*} = \bigcap_{v \in \mathbf{V}_\infty \setminus \{0\}} N_{\langle v \rangle}^{fin}$ . Clearly  $N_{\omega^*} \subset \bigcap_{v \in \mathbf{V}_\infty \setminus \{0\}} N_{\langle v \rangle}^{fin}$ . To show the other inclusion, note that if  $< \in X_{\mathcal{K}^*}$  with  $< / \sim \neq \omega^*$ , then there are  $[u], [v_i] \in \mathbf{V}_\infty / \sim$ ,  $i \in \mathbb{N}$ , with  $[v_i] \neq [v_j]$  if  $i \neq j$  and  $[u] < [v_i]$  for each  $i$ . Pick  $u \in [u]$ . Then  $< \notin N_{\langle u \rangle}^{fin}$ .  $\square$

## 6 Matrices of Ordered Inclusion

In this section we develop some of the tools we will need to prove Theorem 7.3 below. Define a *chain* to be a sequence of subspaces  $V_1 \subset V_2 \subset \dots$  with  $V_n$   $n$ -dimensional and  $\bigcup_{m \geq 1} V_m = \mathbf{V}_\infty$ . Given a chain and an ordering  $< \in X_{\mathcal{K}^*}$ , we will write  $<_n = <|_{V_n}$ . We will write  $B_n = \{b_0^{(n)} > \dots > b_{n-1}^{(n)}\}$  for the least basis of  $<_n$  in  $V_n$ , i.e. the basis which induces the antilexicographic ordering.

Let  $(V_i)_{i \in \mathbb{N}}$  be a chain and  $< \in X_{\mathcal{K}^*}$ . For  $m > n$ , we may represent the inclusion map  $i : \langle V_n, <_n \rangle \hookrightarrow \langle V_m, <_m \rangle$  via the change of basis matrix  $M_{n,m}$ ; writing  $M_{n,m} = (m_{ij})$ ,  $0 \leq i < m$ ,  $0 \leq j < n$ , we have:

$$b_j^{(n)} = \sum_{i=0}^{n-1} m_{ij} b_i^{(m)}$$

We will call  $M_{n,m}$  as above the *matrix of ordered inclusion*; we will use the shorthand  $M_n = M_{n,n+1}$  when there is no confusion. We see that  $M_{n,m} = M_{m-1} \dots M_n$ . This leads us to ask the following:

**Question 6.1.** *For which  $m \times n$  matrices  $M$  is there  $< \in X_{\mathcal{K}^*}$  such that  $M = M_{n,m}$ ?*

First observe that if  $< \in X_{\mathcal{K}^*}$ , then the following must hold:

1. For  $i < j$ , we have  $b_j^{(n)} <_m b_i^{(n)}$ .
2. For any  $b_i^{(n)} \in B_n$ , we have  $\forall u \in V_n \left( u <_m b_i^{(n)} \Rightarrow u \ll_m b_i^{(n)} \right)$ .

**Proposition 6.2.** *Let  $B_n, B_m$  be any ordered bases of  $V_n, V_m$  inducing orderings  $<_n, <_m$ . Let  $M$  be the matrix of inclusion with respect to these bases. If (1) and (2) hold, then  $<_m$  extends  $<_n$ .*

*Proof.* Let  $Z = \{z_0 > \dots > z_{n-1}\}$  be the least basis of  $\langle V_n, <_m |_{V_n} \rangle$ . Consider the following subset of  $U$ :

$$X := \{v \in U : \forall u \in U (u <_m v \Rightarrow u \ll_m v)\}$$

We see that  $Z \subseteq X$  and  $B_n \subseteq X$ . However, we also see that for  $x_1 \neq x_2 \in X$ ,  $x_1 \not\prec x_2$ . Hence  $|B| \leq n$ , and  $Z = B_n$ . It follows that  $b_i^{(n)} = z_i$ ,  $0 \leq i < n$ .  $\square$

Call an  $m \times n$  matrix  $M$  *valid* if  $M = M_{n,m}$  for some  $< \in X_{\mathcal{K}^*}$  and some chain. Write  $M = (m_{ij})$ ,  $0 \leq i < m$ ,  $0 \leq j < n$ . For each column  $j$  of  $M$ , let  $m_j$  denote the least row number with  $m_{m_j j}$  nonzero. If  $M$  is valid, witnessed by  $< \in X_{\mathcal{K}^*}$  and some chain, then the following must hold:

- For  $k < l$ , we have  $m_k < m_l$  since  $b_l^{(n)} \ll_m b_k^{(n)}$ .
- For each  $k$ , we have  $m_{m_k k} = 1$ , as each  $b \in B_n$  is minimal in its line.
- For  $l \neq k$ , we have  $m_{m_k l} = 0$ . For  $l > k$ , this is clear. For  $l < k$ , it is because we have  $b_l^{(n)} <_m b_l^{(n)} + c b_k^{(n)}$  for all  $c \in F_q \setminus \{0\}$ .

The necessary conditions amount to saying that  $M$  is the transpose of a matrix in reduced row echelon form with rank  $n$ . These conditions are also sufficient; let  $M$  satisfy the above, and let  $(V_i)_{i \in \mathbb{N}}$  be a chain. Fix an ordered basis  $B_n$  of  $V_n$ , and choose an ordered basis  $B_m$  such that  $M$  is the matrix of inclusion with respect to these bases; we may do this as  $M$  has rank  $n$ . We will show that the two conditions of Proposition 6.2 are satisfied. The first is clear. Now, for  $b_i^{(n)} \in B_n$ , suppose  $u \in V_n$  with  $u <_m b_i^{(n)}$ . Write  $u = u_0 b_0^{(n)} + \dots + u_{n-1} b_{n-1}^{(n)}$ , and suppose  $k$  is least with  $u_k \neq 0$ ; we are done if we can show  $k > i$ . In the basis  $B_m$ , we have:

$$\begin{aligned} u &= \sum_{j=0}^{n-1} \left( u_j \left( \sum_{i=0}^{m-1} m_{ij} b_i^{(m)} \right) \right) \\ &= \mu_0 b_0^{(m)} + \dots + \mu_{m-1} b_{m-1}^{(m)} \end{aligned}$$

We see that  $m_k$  is least with  $\mu_{m_k} \neq 0$ , so we must have  $k \geq i$ . Suppose  $k = i$ . Then  $u_k = 1$ . Let

$$\begin{aligned} x = u - b_i^{(n)} &= x_0 b_0^{(n)} + \dots + x_{n-1} b_{n-1}^{(n)} \\ &= \chi_0 b_0^{(m)} + \dots + \chi_{m-1} b_{m-1}^{(m)} \end{aligned}$$

and let  $j$  be least with  $x_j \neq 0$ . We have that  $j > i$ . Then  $m_j$  is least with  $\chi_{m_j} \neq 0$ . But  $m_{m_j i} = 0$ , from which it follows that  $u >_m b_i^{(n)}$ , a contradiction.

The case  $m = n + 1$  will be of special interest, so let us introduce some terminology specific to this case. For  $M$  a valid matrix, the map  $i \rightarrow m_i$  has range which excludes a single number  $k$ ,  $0 \leq k \leq n$ . Call such a matrix type  $k$ . Denote the type of matrix  $M$  by  $t(M)$ . A natural question to ask is how many valid  $(n + 1) \times n$  matrices  $M$  have  $t(M) = k$ . By the necessary conditions above, we see that  $m_{ij}$  is determined except for those pairs  $(i, j)$  with both  $i = k$  and  $j < k$ ; for these values of  $i$  and  $j$ , any choice of  $m_{ij} \in F_q$  gives us a valid  $M$ . Hence there are  $q^k$  valid matrices  $M$  of type  $k$ . Observe that  $[V_n, <_n]_k \subset [V_{n+1}, <_{n+1}]_k$  if  $t(M_n) > k$ , and  $[V_n, <_n]_k \subset [V_{n+1}, <_{n+1}]_{k+1}$  if  $t(M_n) \leq k$ . In particular,  $t(M_n) = k$  iff  $[V_{n+1}, <_{n+1}]_k \cap V_n = \emptyset$ .

Let  $\mathcal{M}_n$  be the set of valid  $(n + 1) \times n$  matrices, which we will equip with the uniform probability measure  $\rho_n$ . Let  $\mathcal{M} = \prod_{n \in \mathbb{N}} \mathcal{M}_n$ , and let  $\rho$  be the product measure. Fix a chain  $(V_i)_{i \in \mathbb{N}}$ . There is a surjection  $\pi : X_{\mathcal{K}^*} \rightarrow \mathcal{M}$  as follows; for any  $< \in X_{\mathcal{K}^*}$ , let  $\pi(<) = (M_i)_{i \in \mathbb{N}}$ .

**Proposition 6.3.** *Let  $\mu$  be the unique measure on  $X_{\mathcal{K}^*}$ . Then  $\rho = \pi_* \mu$ .*

*Proof.* First we will show the following: let  $<_n$  be a natural order on  $V_n$ , and let  $M \in \mathcal{M}_n$ . Then I claim that the number of natural  $<_{n+1}$  on  $V_{n+1}$  extending  $<_n$  such that the matrix of ordered inclusion is  $M$  does not depend on  $M$  or  $<_n$ . Write  $M = (m_{ij})$ ,  $0 \leq i < n + 1$ ,  $0 \leq j < n$ . Pick  $n$  linearly independent rows of  $M$ ; without loss of generality assume rows  $0 \leq i < n$  are independent. We now have the following procedure for producing  $<_{n+1}$  extending  $<_n$ : first pick  $b_n^{(n+1)}$  from among the  $q^{n+1} - q^n$  vectors in  $V_{n+1} \setminus V_n$ . Now set the other  $b_i^{(n+1)}$  to be the unique solution to the system of equations:

$$\sum_{i=0}^{n-1} m_{ij} b_i^{(n+1)} = b_j^{(n)} - m_{nj} b_n^{(n+1)} \quad (0 \leq j < n)$$

This procedure can produce any natural  $<_{n+1}$  extending  $<_n$  with matrix of ordered inclusion  $M$ . Moreover, we see that there are  $q^{n+1} - q^n$  such extensions.

Let  $U(S_1, \dots, S_k) = S_1 \times \dots \times S_k \times \prod_{n > k} \mathcal{M}_n$  be a basic open set in  $\mathcal{M}$ . Note that whether or not an ordering  $<$  is in  $\pi^{-1}(U)$  depends only on  $<|_{V_{k+1}}$ . Now we have:

$$\begin{aligned} \pi_* \mu(U) &= \pi_* \mu(U(S_1) \cap U(\mathcal{M}_1, S_2) \cap \dots \cap U(\mathcal{M}_1, \dots, \mathcal{M}_{k-1}, S_k)) \\ &= \pi_* \mu(U(S_1)) * \pi_* \mu(U(\mathcal{M}_1, S_2)) * \dots * \pi_* \mu(U(\mathcal{M}_1, \dots, \mathcal{M}_{k-1}, S_k)) \\ &= (|S_1|/|\mathcal{M}_1|) * \dots * (|S_k|/|\mathcal{M}_k|) \\ &= \rho(U(S_1, \dots, S_k)) \end{aligned}$$

□

## 7 A Representation of $\mu$

To conclude this paper, we provide a more concrete representation of the measure  $\mu$ . First, we need a few general lemmas.

**Lemma 7.1.** *Let  $\langle V, < \rangle$  have the FLP, and fix  $v \in V$ . Suppose  $u, w \in V$  are minimal in their lines with  $u \sim w$ . Let  $c_u \in F_q$  be such that  $v - c_u u < v - du$  for  $d \neq c_u$ , and likewise for  $w$ . Then  $c_u = c_w$ .*

*Proof.* Form  $U = \langle u, v, w \rangle$ , and let  $<|_U$  be induced by basis  $\{x_0, \dots, x_l\}$ ,  $l \leq 2$ . Write  $u = u_0 x_0 + \dots + u_l x_l$ , etc. Let  $k$  be the least number with  $u_k$  nonzero. Then  $k$  is also the least number with  $w_k$  nonzero. We have  $u_k = w_k = 1$ ; we see that  $c_u = c_w = v_k$ .  $\square$

**Lemma 7.2.** *Let  $w \in V$  be minimal in its line. For any  $u \in V$ , let  $c_u \in F_q$  be such that  $u - c_u w < u - dw$  for  $d \neq c_u$  (this is a slight change in notation from Lemma 7.1). Then  $c_{u+v} = c_u + c_v$  and  $c_{du} = dc_u$ ,  $d \in F_q$ .*

*Proof.* Let  $u, v \in V$ . Form  $U = \langle u, v, w \rangle$ , and let  $<|_U$  be induced by basis  $\{x_0, \dots, x_l\}$ ,  $l \leq 2$ . Write  $u = u_0 x_0 + \dots + u_l x_l$ , etc. Let  $k$  be the least number with  $w_k$  nonzero. Then  $w_k = 1$ ; we see that  $c_u = u_k$ ,  $c_v = v_k$ ,  $c_{u+v} = u + v_k = u_k + v_k$ , and  $c_{du} = dc_u = du_k$ .  $\square$

We may now injectively map  $\langle V, < \rangle$  to a subspace of  $\langle F_q^{V/\sim}, \prec \rangle$ , where  $\prec$  is a partial ordering with  $\alpha \prec \beta$  iff for some  $[u]$  in  $V/\sim$ , we have  $\alpha([u]) < \beta([u])$  and  $\alpha([v]) \leq \beta([v])$  for all  $[v] > [u]$ . We will be most interested in the case  $V/\sim = \omega^*$ ; in this case  $\prec$  is a linear order, as  $V/\sim$  has no infinite ascending chains. For  $< \in N_{\omega^*}$ , call the equivalence classes of  $\mathbf{V}_{\infty}$   $\alpha_0 > \alpha_1 > \dots$ . Now for any  $< \in N_{\omega^*}$  and  $v \in \mathbf{V}_{\infty}$ , we may identify  $v \in F_q^{\omega^*}$ . To make this identification explicit, pick  $w_i \in \alpha_i$  minimal in their lines, and define  $\phi_{<} : \langle \mathbf{V}_{\infty}, < \rangle \rightarrow \langle F_q^{\omega^*}, \prec \rangle$  via  $\phi_{<}(v) = (v_i)_{i \in \omega^*}$ , where  $v_i$  is such that  $v - v_i w_i \leq v - dw_i$  for all  $d \in F_q$ . Note that  $\phi_{<}$  does not depend on the choice of  $w_i$ .

Fix any basis  $B = \{b_0, b_1, \dots\}$  of  $\mathbf{V}_{\infty}$ . Define  $\phi : N_{\omega^*} \rightarrow (F_q^{\omega^*})^{\omega^*}$  by setting  $\phi(<) = (\phi_{<}(b_i))_{i \in \omega^*}$ . Note that  $\phi$  is a Borel map. Equip  $(F_q^{\omega^*})^{\omega^*}$  with the product measure  $\sigma$ .

**Theorem 7.3.** *The map  $\phi$  is injective and a.e. surjective. Moreover,  $\sigma = \phi_* \mu$ , giving a mod zero isomorphism of  $(X_{\mathcal{K}^*}, \mu)$  and  $((F_q^{\omega^*})^{\omega^*}, \sigma)$ .*

*Proof.* To see injectivity, it suffices to note that for any  $\beta \in \text{Im}(\phi)$  and any  $< \in \phi^{-1}(\beta)$ , the map  $\phi_{<}$  is completely determined, which in turn determines  $<$ .

To show that  $\phi$  is a.e. surjective, consider  $\beta = (\beta_i)_{i \in \omega^*} \in (F_q^{\omega^*})^{\omega^*}$ . Certainly  $\beta \in \text{Im}(\phi)$  if the following hold:

1. The  $\beta_i$  are linearly independent.
2. For each  $k > 0$ , there is an  $i$  with  $\beta_i|_k = 0 \wedge \dots \wedge 0 \wedge c$ ,  $c \neq 0$ .

The second condition is easily seen to be the countable intersection of measure 1 conditions. For the first condition, observe that this is the countable intersection of conditions  $c_0 \beta_{i_0} + \dots + c_k \beta_{i_k} \neq 0$ , each of which is measure 1.  $\square$

To show that  $\sigma = \phi_*\mu$ , it suffices to prove the next lemma. For  $v_0, \dots, v_{n-1} \in \mathbf{V}_\infty$  and  $s_0, \dots, s_{n-1} \in F_q^k$ , define:

$$N(v_0, s_0, \dots, v_{n-1}, s_{n-1}) = \{< \in N_{\omega^*} : v_i = s_i \hat{\ } \beta_i \text{ for some } \beta_i \in F_q^{\omega^*}\}$$

**Lemma 7.4.** *For  $v_0, \dots, v_{n-1}$  linearly independent,  $\mu(N(v_0, s_0, \dots, v_{n-1}, s_{n-1})) = q^{-kn}$ .*

*Proof.* Fix  $< \in N_{\omega^*}$  and a chain  $(V_i)_{i \in \mathbb{N}}$ , and let  $\pi$  be as in Proposition 5. First we consider the case where the  $s_i$  are linearly independent. The probability that  $v_i = s_i \hat{\ } \beta_i$  for  $0 \leq i < n$  is bounded below by the probability that the following both occur for some  $l$  with  $v_0, \dots, v_{n-1} \in V_l$ :

1. Let  $<_l$  on  $V_l$  be given by basis  $\{x_0, \dots, x_{l-1}\}$ . Write  $v_i = a_0^i x_0 + \dots + a_{l-1}^i x_{l-1}$ . Then  $a_0^i \hat{\ } \dots \hat{\ } a_{k-1}^i = s_i$ .
2. Now suppose  $\pi(<) = (M_i)_{i \in \mathbb{N}} \in \mathcal{M}$ . Then we have  $t(M_i) \geq k$  for  $i \geq l$ .

Call the first event  $A_1(l)$  and the second event  $A_2(l)$ ; these events are independent. For linearly independent  $(a_0^i, \dots, a_{l-1}^i)$ ,  $0 \leq i < n$ , the number of ordered bases  $Y = \{y_0, \dots, y_{l-1}\}$  with  $v_i = a_0^i y_0 + \dots + a_{l-1}^i y_{l-1}$ ,  $0 \leq i < n$ , does not depend on which particular linearly independent  $(a_0^i, \dots, a_{l-1}^i)$  are being considered. Therefore:

$$\begin{aligned} \mathbf{P}(A_1(l)) &= \frac{\#(\text{linearly independent } (a_0^i, \dots, a_{l-1}^i) \text{ with } a_0^i \hat{\ } \dots \hat{\ } a_{k-1}^i = s_i)}{\#(\text{linearly independent } (a_0^i, \dots, a_{l-1}^i))} \\ &= \frac{q^{n(l-k)}}{(q^l - 1)(q^l - q) \dots (q^l - q^{n-1})} \end{aligned}$$

Now for  $i \geq l$ , let  $B_i(k)$  be the event that  $t(M_i) \geq k$ . We have  $\mathbf{P}(B_i(k)) = \frac{q^{i+1} - q^k}{q^{i+1} - 1}$ . The events  $B_i(k)$  are mutually independent, hence:

$$\begin{aligned} \mathbf{P}(A_2(l)) &= \lim_{m \rightarrow \infty} \prod_{l \leq i < m} \mathbf{P}(B_i(k)) \\ &= \lim_{m \rightarrow \infty} \frac{q^{(m-l)(k)} (q^{l-k+1} - 1) \dots (q^l - 1)}{(q^{m-k+1} - 1) \dots (q^m - 1)} \\ &\geq (1 - q^{k-1-l})^k \end{aligned}$$

Now we have:

$$\lim_{l \rightarrow \infty} \mathbf{P}(A_1(l)) \cdot \mathbf{P}(A_2(l)) = q^{-nk}$$

It follows that  $\mu(N(v_0, s_0, \dots, v_{n-1}, s_{n-1})) \geq q^{-nk}$ . Equality follows since for  $(t_0, \dots, t_{n-1}) \neq (s_0, \dots, s_{n-1})$ , we have  $N(v_0, s_0, \dots, v_{n-1}, s_{n-1}) \cap N(v_0, t_0, \dots, v_{n-1}, t_{n-1}) = \emptyset$ .

When the  $s_i$  are not linearly independent, let:

$$L^m = \{(t_0, \dots, t_{n-1}) : t_i \in F_q^m \text{ and the } t_i \text{ are linearly independent}\}$$

We have the lower bound:

$$\mu(N(v_0, s_0, \dots, v_{n-1}, s_{n-1})) \geq \sum_{(t_0, \dots, t_{n-1}) \in L^m} N(v_0, s_0 \wedge t_0, \dots, v_{n-1}, s_{n-1} \wedge t_{n-1})$$

We have  $|L^m| = (q^m - 1)(q^m - q) \dots (q^m - q^{n-1})$ , giving us:

$$\mu(N(v_0, s_0, \dots, v_{n-1}, s_{n-1})) \geq q^{-k(n+m)}(q^m - 1) \dots (q^m - q^{n-1})$$

for any  $m$ . Letting  $m \rightarrow \infty$ , we see that  $\mu(N(v_0, s_0, \dots, v_{n-1}, s_{n-1})) \geq q^{-nk}$ , and hence  $\mu(N(v_0, s_0, \dots, v_{n-1}, s_{n-1})) = q^{-nk}$ . □

## 8 Questions and Further Work

Our investigations above lend themselves to a number of open questions:

**Question 8.1.** *Are there other examples where Theorem 2.1 can be used to show non-amenability? Are there any examples where Theorem 2.1 can be used to show amenability?*

**Question 8.2.** *Assume the Fraïssé class  $\mathcal{K}$  admits a companion  $\mathcal{K}^*$ . Let  $\mathbf{K}$  be the Fraïssé limit of  $\mathcal{K}$ . If  $\text{Aut}(\mathbf{K})$  is uniquely ergodic, what are necessary and sufficient conditions for the unique measure on any minimal flow to be supported on the generic orbit?*

Pongrácz in [P] has given a partial answer to Question 8.2. Let  $L^* = L \cup \{<\}$ , for  $<$  a symbol for a linear ordering and  $L$  **relational**. Let  $(\mathcal{K}, \mathcal{K}^*)$  be an excellent pair of Fraïssé classes in  $L$  and  $L^*$ . Suppose  $\mathcal{K}^*$  is order forgetful, i.e. for  $\langle \mathbf{A}, < \rangle, \langle \mathbf{B}, <' \rangle \in \mathcal{K}^*$ , we have  $\mathbf{A} \cong \mathbf{B} \Leftrightarrow \langle \mathbf{A}, < \rangle \cong \langle \mathbf{B}, <' \rangle$ . We see that  $\text{Aut}(\mathbf{K})$ , if amenable, is uniquely ergodic, and the measure satisfies  $\mu(N_{\langle \mathbf{A}, < \rangle}) = 1/k_{\mathbf{A}}$  (see the introduction). Pongrácz has shown that in this case,  $\mu$  is supported generically. Note that this does not contradict Theorem 1.2; every hypothesis of Pongrácz's theorem is satisfied except that the language of vector spaces contains function symbols. His calculations also shed some light on the role that functions play in my calculations for  $\mathbf{V}_\infty$ , and they also suggest that we may be able to find relational examples with  $L^* = L \cup \{S_1, \dots, S_n\}$  with the measure  $\mu$  as above not supported generically.

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